Walsh Functions, Gray Traversals, and Hilbert Curves:

Three Views of Binary Hypercube Geometry

The Fast Walsh–Hadamard Transform, Gray-code traversals, and Hilbert curves are often presented as unrelated constructions arising from signal processing, combinatorics, and geometry respectively. In practice they are deeply connected. All three can be interpreted as different operations on the binary hypercube.

This note develops an intuitive and relatively practical viewpoint linking these structures together.

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1. The Binary Hypercube

Consider all binary vectors of length n:

$x \in \{0,1\}^n$

These vectors form the vertices of an n-dimensional hypercube.

Two vertices are adjacent if they differ in exactly one bit.

For example, for n=3:

$000 \leftrightarrow 001 \leftrightarrow 011 \leftrightarrow 010 \leftrightarrow \cdots$

Many seemingly unrelated binary algorithms are really:

- traversals of the hypercube,
- oscillations on the hypercube,
- or transformations of functions defined on hypercube vertices.

The key idea of this note is:

Structure	Interpretation
Walsh functions	Standing waves on the hypercube
Gray traversals	Minimal-edge walks on the hypercube
Hilbert curves	Recursive locality-preserving embeddings of hypercube traversals

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2. Walsh Functions as Hypercube Oscillations

The Walsh basis functions are:

$$\chi_a(x) = (-1)^{a \cdot x}$$

where:

- a selects a mode,
- x is a hypercube vertex,
- the dot product is modulo 2.

The Walsh function changes sign whenever traversal crosses one of the active dimensions selected by a.

For example:

- a=001: oscillates along the last dimension,
- a=111: oscillates whenever any participating parity changes.

The Hamming weight of a controls how rapidly the mode oscillates across the cube.

Intuitively:

- low-weight modes are smooth,
- high-weight modes alternate rapidly.

The Walsh basis therefore acts as a spectral decomposition of binary structure.

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3. Gray Traversals as Minimal Hypercube Walks

A Gray code orders vertices such that adjacent entries differ by one bit.

For example:

000, 001, 011, 010, 110, 111, 101, 100

This is not merely convenient indexing.

It is a Hamiltonian path through the hypercube.

At each step:

- exactly one dimension changes,
- motion occurs along one hypercube edge.

The binary Gray transform is:

$$g(x) = x \oplus (x \gg 1)$$

This converts ordinary binary ordering into a locality-preserving traversal.

The important connection to Walsh functions is that Gray traversals move through Walsh sign regions smoothly.

Each Gray transition crosses exactly one parity boundary.

Thus Gray traversal acts almost like a “continuous scan” through Walsh phase space.

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4. Walsh Functions Along a Gray Traversal

The connection becomes clearer if Walsh modes are sampled along a Gray-code path.

A Walsh mode becomes a binary square wave whose transitions occur when the Gray walk crosses parity boundaries.

For a Walsh mode indexed by a:

- dimensions selected in a determine where sign changes occur,
- Gray traversal converts multidimensional parity oscillations into 1D switching sequences.

This explains why:

- Walsh functions resemble digital square waves,
- Gray ordering produces low-transition signal representations,
- sequency ordering behaves like binary frequency.

In effect:

> Gray traversal converts hypercube geometry into a one-dimensional scan, while Walsh modes describe oscillatory structure on that same geometry.

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5. Hilbert Curves as Recursive Gray Traversals

The Hilbert curve extends the Gray-code idea spatially.

A Hilbert traversal:

- recursively subdivides space,
- traverses subcells in Gray-like order,
- rotates and reflects local coordinate systems,
- preserves locality much better than simple raster order.

The crucial observation is:

> A Hilbert curve is fundamentally a recursively rotated Gray-code traversal.

At every recursion level:

- only one major spatial transition occurs at a time,
- entry and exit points are coordinated recursively,
- local orientation state propagates downward.

This is why Hilbert curves maintain spatial continuity so effectively.

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6. Gray Traversals and Hilbert Orientation State

Ordinary Gray code alone is insufficient for good multidimensional locality.

Hilbert traversal adds:

- recursive orientation transforms,
- coordinate-frame transport,
- dimension permutations,
- conditional reflections.

Conceptually:

Mechanism	**Purpose**
Gray transition	Minimize local movement
Recursive subdivision	Multiscale locality
Orientation updates	Maintain continuity
Reflections/permutations	Match entry and exit geometry

Thus the Hilbert curve can be viewed as:

- a Gray traversal,
- plus recursively transported coordinate frames.

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7. Shared Binary Operations

All three systems rely heavily on the same binary primitives:

Primitive	**Walsh**	**Gray**	**Hilbert**
XOR	Core algebra	Bit transitions	Orientation logic

Parity	Sign changes	Transition selection	Rotation state
Bit permutations	Sequency ordering	Traversal order	Coordinate transforms
Recursive subdivision	Butterfly stages	Recursive Gray generation	Recursive geometry
Hypercube edges	Mode oscillation	Path traversal	Embedded traversal

This commonality explains why implementations often reuse the same low-level operations:

- XOR,
- prefix XOR,
- bit rotations,
- Gray transforms,
- bit shuffles,
- signed permutations.

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8. The FWHT Butterfly as Hypercube Propagation

The Fast Walsh–Hadamard Transform performs recursive pairwise mixing:

$(a,b) \rightarrow (a+b, a-b)$

Each stage mixes vertices differing in one bit.

Thus:

- stage 0 mixes neighbors along dimension 0,
- stage 1 mixes along dimension 1,
- etc.

The butterfly network is therefore a structured propagation process over the hypercube graph.

This creates another link:

Object	Hypercube Interpretation
FWHT butterfly	Recursive diffusion
Gray traversal	Edge-following path
Hilbert traversal	Locality-preserving embedding

All are different ways of organizing information flow across binary adjacency relations.

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9. Locality and Correlation

All three constructions are fundamentally trying to manage locality.

Walsh functions:

- organize correlation structure spectrally.

Gray traversals:

- minimize transition distance.

Hilbert curves:

- preserve geometric neighborhoods after flattening.

This makes them highly useful for:

- caches,
- memory layouts,
- associative memory,
- sparse neural attention,
- multidimensional indexing,
- binary embeddings,
- hyperdimensional computing.

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10. A Unified View

The three constructions can be summarized as follows:

Structure	Primary Role
Walsh basis	Spectral decomposition
Gray traversal	Continuous hypercube traversal
Hilbert curve	Spatial embedding of traversal

Or more intuitively:

- Walsh functions describe oscillations on the hypercube.
- Gray codes describe smooth motion on the hypercube.
- Hilbert curves embed that motion into Euclidean space while preserving locality.

These are not isolated mathematical curiosities. They form a coherent family of binary recursive structures.

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11. Practical Implications

This unified viewpoint suggests several practical directions.

Fast structured neural mixing

FWHT layers:

- efficiently spread information globally,
- preserve binary structure,
- require only O(nlogn) operations.

Gray/Hilbert orderings:

- improve locality,
- reduce memory movement,
- support sparse structured processing.

Together they suggest efficient binary-native architectures.

Associative memory

Gray and Hilbert orderings:

- cluster nearby states,
- preserve partial similarity.

Walsh spectra:

- expose parity correlations,
- identify distributed structure.

This combination may help construct scalable distributed associative memories.

Sparse attention and token layouts

Hilbert orderings can:

- linearize multidimensional data,
- preserve locality,
- reduce attention bandwidth.

Walsh mixing can:

- rapidly spread information globally,
- create structured long-range interactions.

This combination is increasingly relevant for efficient transformer architectures.

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12. Conclusion

Walsh functions, Gray traversals, and Hilbert curves are best understood not as isolated algorithms, but as different manifestations of binary hypercube geometry.

- Walsh functions define spectral modes.
- Gray codes define smooth traversals.
- Hilbert curves define locality-preserving embeddings.

All three rely on:

- recursive subdivision,
- parity structure,

- XOR algebra,
- and hypercube adjacency.

Viewed this way, many classical “bit tricks” become components of a much larger geometric and computational framework.